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From measurements to quantum friction

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Abstract

We present a quantum theory of friction in which interactions with the surrounding medium are described by generalized measurements of the particle's position and momentum. The theory predicts intrinsically quantum contributions to the particle's steady-state energy and to the associated diffusion in position. We discuss the physical significance of these and demonstrate their significance in ensuring a well behaved theory.

1. Introduction

The simplest way to include dissipation in a mechanical model is to add a force which is proportional to the velocity of the damped particle with the sign chosen so as to oppose the motion. This procedure is commonly taught in elementary mechanics courses, notably in treating the damped harmonic oscillator [1]. For an otherwise free particle, this resistive force damps out the motion, leaving the particle stationary in the long-time limit. The resulting zero kinetic energy is at odds with thermodynamic equilibrium, which suggests that the particle should retain, on average, an energy of $k_{\mathrm{B}}T/2$ per degree of freedom. We can resolve this problem by introducing a random fluctuating force into our equations of motion [2]. This leads to a fluctuating Brownian motion of the damped particle, the statistics of which depend only on the viscosity and temperature of the surrounding medium and on the mass of the particle.

It is surprising to note, however, that 100 years after Einstein's classic work on Brownian motion [3] the quantum description of friction remains a problem. Mapping the classical equations directly onto the quantum theory leads to equations of motion that fail to preserve the positivity of probabilities [4–6]. A microscopic model based on coupling to a bath of harmonic oscillators has also been proposed [7, 8]. The resulting dynamics is well behaved but appears to be intrinsically non-Markovian in that the evolution of the density operator depends on its history rather than just on its present form [9]. This feature is a significant

[3] I would like to dedicate this paper to Marshall Stoneham on the occasion of his 65th birthday. I spent only one year working with Marshall but, looking back, it was probably the most significant of my career.

complication compared to the simple Markovian dynamics that suffice to describe the classical dynamics. Dissipative equations for systems with a well defined oscillation frequency are very well understood, of course, and have been an important component of quantum optics research for many years [10–12]. Systems without this frequency, like the otherwise free particle undergoing Brownian motion, can respond to a very wide range of reservoir frequencies, and this is the origin of the non-Markovian behaviour. A microscopic approach to friction has been proposed in which the individual collisions between the particle of interest and molecules of the surrounding medium are modelled in detail [13–18]. One interesting and early approach was to treat the centre of mass motion of the particle classically but subject it to collisional dynamics derived from quantum theory [19]. More recently, it has been noted that continuously monitoring the particle's position can lead to motion characteristic of a Brownian particle [20]. These approaches and others are discussed at length in two recent books [21, 22] and in the introduction to our recent paper [23]. We present a short overview of the quantum friction problem in the next section of this paper.

Our approach has been to seek a simple quantum description of friction which obeys the rules of quantum theory but at the same time is a simple as possible, so as to retain the simplicity of classical treatments [23]. The physical principle we follow is that each collision with a molecule of the medium transfers information on both the position and momentum of our particle to the medium and hence can be treated as a measurement. Simultaneous measurements of position and momentum are perfectly possible, of course, as long as they are not too precise [24–26] and the information imparted to an individual molecule would be very imprecise. Section 3 presents a description of such *generalized measurements* and how we describe a random sequence of them. In section 4 we present our master equation for friction and analyse the dynamics associated with it. Our aim in writing this paper is to present a more extensive discussion of the physical nature and consequences of our model than was possible in our earlier paper [23]. Throughout the paper we consider only a *single spatial dimension* so as to avoid unnecessary complexity of notation and to keep the problem as simple as possible.

2. The friction problem

The simplest method for introducing friction into classical mechanics is to postulate a force that is proportional to and opposes a particle's velocity or momentum:

$$F = -\gamma p, \tag{1}$$

where γ is the frictional coefficient, related to the viscosity of the surrounding medium. In the absence of other forces, this leads to a steady state in which the particle is stationary: $p(\infty) = 0$. This is at odds with the requirements of thermal equilibrium, which state that

$$\overline{p^2(\infty)} = Mk_{\mathrm{B}}T, \tag{2}$$

where M is the mass of the damped particle. One very simple and elegant way to restore equilibrium is to add to (1) a randomly fluctuation force $F(t)$ with the average properties

$$\begin{aligned} &\overline{F(t)} = 0 \\ &\overline{F(t)F(t')} = 2M\gamma k_{\mathrm{B}}T\delta(t-t'). \end{aligned} \tag{3}$$

These correlation functions reflect both the randomness of this force and also its Markovian character, in that the instantaneous value of the force is independent of its value at any other time. Averaging the equations of motion leads to equations for the evolution of the statistical

moments of the particle's position and momentum, the simplest of which are

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\overline{x} &= \frac{\overline{p}}{M} \\
\frac{\mathrm{d}}{\mathrm{d}t}\overline{p} &= -\gamma\overline{p} \\
\frac{\mathrm{d}}{\mathrm{d}t}\overline{x^2} &= 2\frac{\overline{xp}}{M} \\
\frac{\mathrm{d}}{\mathrm{d}t}\overline{xp} &= \frac{\overline{p^2}}{M} - \gamma\overline{xp} \\
\frac{\mathrm{d}}{\mathrm{d}t}\overline{p^2} &= -2\gamma\overline{p^2} + 2M\gamma k_{\mathrm{B}}T.
\end{aligned} \tag{4}$$

The solution of these is clearly consistent with the requirements for steady-state kinetic energy (2). It also reproduces the famous steady-state Brownian diffusion of the particle's position [3]:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\overline{x^2}\right|_{t\to\infty} = \frac{2\overline{p^2(\infty)}}{M^2\gamma} = \frac{2k_{\mathrm{B}}T}{M\gamma}. \tag{5}$$

In quantum theory, the statistical properties of our particle are described by its density operator $\hat{\rho}$ [27], the evolution of which is determined by a master equation [10–12]. The most natural approach to producing a quantum theory of friction would be to write down a master equation that gives, as quantum expectation values, the equations (4). The resulting master equation has the form [5, 6, 21, 22]

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}\left[\frac{\hat{p}^2}{2M},\hat{\rho}\right] - \mathrm{i}\frac{\gamma}{2\hbar}\left[\hat{x},\left\{\hat{p},\hat{\rho}\right\}\right] - \frac{M\gamma k_{\mathrm{B}}T}{\hbar^2}\left[\hat{x},\left[\hat{x},\hat{\rho}\right]\right], \tag{6}$$

where $[\cdots,\cdots]$ and $\{\cdots,\cdots\}$ denote a commutator and an anti-commutator respectively. This equation conserves the total probability, in that $\mathrm{Tr}\hat{\rho} = 1$, but the probabilities associated with individual states can become greater than unity while others become negative. We can illustrate this unfortunate state of affairs by considering an initial pure state, $\hat{\rho} = |\psi\rangle\langle\psi|$, and calculating the initial rate of change of the probability for remaining in that state [5, 6]:

$$\langle\psi|\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t}|\psi\rangle\Big|_{t=0} = \frac{\gamma}{2} - \frac{2\gamma Mk_{\mathrm{B}}T}{\hbar^2}\Delta x^2(0). \tag{7}$$

The initial probability for being in the initial state is, of course, unity, and as this is the maximum possible value that is physically allowed the derivative should be negative. If, however, the particle is initially very well localized, so that

$$\Delta x(0) < \frac{\hbar}{2\sqrt{Mk_{\mathrm{B}}T}} = \frac{\lambda_{\mathrm{dB}}^T}{4\pi}, \tag{8}$$

then the probability for remaining in the initial state *grows* at short times and so exceeds unity. The condition (8) corresponds to initially localizing the particle well within a thermal de Broglie wavelength (λ_{dB}^T). Classical mechanics emerges, of course, on length scales greatly in excess of the de Broglie wavelength so that $\Delta x(0) \gg \lambda_{\mathrm{dB}}^T$. We can infer, therefore, that the problem with the master equation (6) is that we have failed to take account of an effect that is intrinsically quantum in origin.

We shall see that the missing phenomenon is a diffusion of the particle position associated with a double momentum commutator in the master equation [20, 23]. In our theory, this arises as a consequence of treating individual collisions as simultaneous measurements of the Brownian particle's position and momentum.

3. Generalized measurements

The conventional view of a measurement is exemplified by von Neumann's treatment in his famous book [28]. There he describes 'two fundamentally different types of interventions which can occur in a system': these are (i) measurements and (ii) the automatic changes which occur with the passage of time. In modern notation, these changes are (i)

$$\hat{\rho} \to \sum_n \langle n|\hat{\rho}|n\rangle |n\rangle\langle n|, \tag{9}$$

where $|n\rangle$ is the nth eigenvector of the measured observable. The probability that the measurement outcome is 'n' is simply $\langle n|\hat{\rho}|n\rangle$, and if this result is recorded then after the measurement the system is left in the corresponding eigenstate $|n\rangle$. The other change, (ii), is simply the evolution associated with the Schrödinger equation:

$$\hat{\rho}(0) \to \hat{\rho}(t) = \exp\left(-\mathrm{i}\hat{H}t/\hbar\right)\hat{\rho}(0)\exp\left(\mathrm{i}\hat{H}t/\hbar\right). \tag{10}$$

The above description of a measurement, however, is insufficiently general for many purposes. Firstly it makes no allowance for the imperfect and often very destructive nature of many measurement processes; the detection of individual photons, for example, is at most about 70% efficient and works by absorbing them so that the number of photons after measurement is certainly not the same as the number measured. Secondly, it does not include the possibility of deliberately imprecise measurements that do not correspond to a single quantum observable. Simplest amongst these is the simultaneous [24, 25] and imprecise or unsharp [26] measurement of a pair of incompatible observables such as position and momentum. These are best understood within the general framework of generalized measurements.

We can generalize von Neumann's description of a measurement by introducing a set of probability operators, $\{\hat{\pi}_i\}$, from which we calculate the probability that the measurement gave the result 'i':

$$P(i) = \mathrm{Tr}\left(\hat{\pi}_i\hat{\rho}\right). \tag{11}$$

This set of operators is known as a probability operator measure [29] or positive operator-valued measure [26, 30]. The probability operators are required to be Hermitian and positive and to resolve the identity operator in that $\sum_i \hat{\pi}_i = \hat{\mathrm{I}}$; these properties they have in common with the projectors $|n\rangle\langle n|$. The difference, however, is that the probability operators are not required to be orthogonal so that we can have $\hat{\pi}_i\hat{\pi}_j \neq \delta_{ij}\hat{\pi}_i$ and hence they need not be projectors. It remains to explain the way in which a generalized measurement changes the density operator. This is most naturally described in terms of an effect [31], that is pairs of Kraus operators $\hat{A}_i$ and $\hat{A}_i^\dagger$. These are related to the probability operator by

$$\hat{\pi}_i = \hat{A}_i^\dagger\hat{A}_i. \tag{12}$$

The generalization of von Neumann's expression (9) can then be written as

$$\hat{\rho} \to \sum_i \hat{A}_i\rho\hat{A}_i^\dagger. \tag{13}$$

Our aim is to produce a master equation for our particle subjected to frequent collisions and to treat these collisions as generalized measurements. We shall assume that these collisions occur instantaneously and randomly at a rate R. We derive the general form of the required master equation by considering the evolution of the density operator in a short time interval Δt. A measurement will occur with probability $R\Delta t$ and the other intervention (Schrödinger evolution) will occur with probability $1 - R\Delta t$. Combining these two possibilities, weighted by their associated probabilities, gives

$$\hat{\rho}(t+\Delta t) = (1 - R\Delta t)\left(\hat{\rho}(t) - \frac{\mathrm{i}}{\hbar}\left[\hat{H}, \hat{\rho}(t)\right]\Delta t\right) + R\Delta t\sum_i \hat{A}_i\rho(t)\hat{A}_i^\dagger, \tag{14}$$

where we have used the short time form of (10). Taking the limit $\Delta t \to 0$ gives the master equation for a system subjected to repeated measurements:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}\left[\hat{H}, \hat{\rho}\right] + R\sum_i \hat{A}_i \hat{\rho} \hat{A}_i^\dagger - R\hat{\rho}. \tag{15}$$

This equation is sufficiently general to describe the evolution of any continuously monitored quantum system. We will discuss its properties in greater detail elsewhere [32].

There are two further generalizations that we can introduce. The first is that a single measurement outcome can be associated with more than one effect operator, so we modify (12) to

$$\hat{\pi}_i = \sum_k \hat{A}_{ik}^\dagger \hat{A}_{ik}. \tag{16}$$

The second, which is of particular relevance to measurements of continuous variables like position and momentum, is that our probability operators and effects can be functions of continuous variables with the summations in the above relations replaced by integrals. For our position and momentum measurements the probability *density* for the results x and p is

$$P(x, p) = \mathrm{Tr}\left(\hat{\pi}(x, p)\hat{\rho}\right), \tag{17}$$

where the probability operator and its associated effect operators satisfy the integral equalities [23]:

$$\begin{aligned} &\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}p\, \hat{\pi}(x, p) = \hat{\mathrm{I}} \\ &\hat{\pi}(x, p) = \int_{-\infty}^{\infty} \mathrm{d}\xi\, \hat{A}_\xi^\dagger(x, p)\hat{A}_\xi(x, p). \end{aligned} \tag{18}$$

Once we have specified the effect operators it is straightforward to obtain the generalization of (15) appropriate for collisions measuring the position and momentum of the particle:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}\left[\hat{H}, \hat{\rho}\right] + R\int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}p\, \hat{A}_\xi(x, p)\hat{\rho}\hat{A}_\xi^\dagger(x, p) - R\hat{\rho}. \tag{19}$$

This is our master equation for friction. In order to use it, however, we need to examine simultaneous position and momentum measurements and to consider what happens in a collision in order to obtain a suitable form for the operators $\hat{A}_\xi(x, p)$.

The simplest procedure is to consider the measurement as a comparison between the density operator $\hat{\rho}$ and a test state. We construct the density operator for the test state by first introducing the Gaussian density operator

$$\hat{\sigma} = \frac{1}{\sqrt{\bar{n}(\bar{n}+1)}} \exp\left[-\ln\left(\frac{\bar{n}+1}{\bar{n}}\right)\frac{1}{2}\left((\hat{p}W/\hbar)^2 + (\hat{x}/W)^2\right)\right]. \tag{20}$$

This state has zero expectation values for both the position and momentum and variances

$$\begin{aligned} \Delta_\sigma x^2 &= W^2(\bar{n} + \tfrac{1}{2}) \\ \Delta_\sigma p^2 &= \frac{\hbar^2}{W^2}\left(\bar{n} + \frac{1}{2}\right). \end{aligned} \tag{21}$$

The notation used in defining σ exploits the analogy between this state and that of a harmonic oscillator with number operator $((\hat{p}W/\hbar)^2 + (\hat{x}/W)^2 - 1)/2$, prepared in a thermal state with mean excitation number $\bar{n}$. We can construct the probability operators from σ by means of a unitary transformation:

$$\hat{\pi}(x, p) = \hat{D}(x, p)\hat{\sigma}\hat{D}^\dagger(x, p), \tag{22}$$

where $\hat{D}(x, p)$ is the Glauber displacement operator [12]

$$\hat{D}(x, p) = \exp\left[\mathrm{i}(p\hat{x} - x\hat{p})/\hbar\right], \tag{23}$$

the action of which moves the expectation values of $\hat{x}$ and $\hat{p}$ from zero to x and p respectively while leaving the variances unchanged. Our measurement consists, therefore, of comparing our state with a continuous set of Gaussian states with the measurement result, (x, p), corresponding to the central coordinates in phase space of the displaced density operator $\hat{\pi}(x, p)$. We could, of course, choose other states as the basis of our probability operators but the Gaussian state has the merits of mathematical simplicity and of depending on a minimum number of parameters (just W and $\bar{n}$). Each measurement gives a value for both the position and the momentum. The average values for the measurement are the expectation values for the state $\hat{\rho}$:

$$\begin{aligned} E(x) &= \mathrm{Tr}(\hat{x}\hat{\rho}) \\ E(p) &= \mathrm{Tr}(\hat{p}\hat{\rho}), \end{aligned} \tag{24}$$

but the associated variances exceed those associated with $\hat{\rho}$:

$$\begin{aligned} \mathrm{Var}(x) &= \Delta_\rho x^2 + \Delta_\sigma x^2 \\ \mathrm{Var}(p) &= \Delta_\rho p^2 + \Delta_\sigma p^2. \end{aligned} \tag{25}$$

This increase, above the intrinsic variances $\Delta_\rho x^2$ and $\Delta_\rho p^2$, is the price we must pay for the ability to measure both the position and momentum [24–26]. For a quantum limited measurement we would require $\Delta_\sigma x \Delta_\sigma p = \hbar/2$, corresponding to $\bar{n} = 0$. We would expect, however, the random collisions associated with Brownian motion to act as very poor measurements so that $\Delta_\sigma x \Delta_\sigma p \gg \hbar/2$, corresponding to $\bar{n} \gg 1$.

Our master equation requires the form of the effect operators $\hat{A}_\xi(x, p)$. The relationship (18) does not, of course, allow us to uniquely determine these from the probability operators $\hat{\pi}(x, p)$. In order to suggest a reasonable form for $\hat{A}_\xi(x, p)$ it is helpful to consider the classical problem. A collision of the Brownian particle, of mass M and momentum p, with an environment particle, of mass m and momentum ζ, causes the momentum to change to

$$p \to \frac{M - m}{M + m} p + \frac{2M}{M + m} \zeta = Kp + \xi, \tag{26}$$

where $K = (M - m)/(M + m)$ and $\xi = (1 + K)\zeta$. The collision occurs essentially instantaneously so that the particle's position is unchanged during the collision. We can build these features into our quantum description of the measurement by setting [23]

$$\hat{A}_\xi(x, p) = \sqrt{\frac{P(\xi)}{2\pi\hbar}}\, \hat{D}(x, Kp + \xi)\hat{\sigma}^{1/2}\hat{D}^\dagger(x, p), \tag{27}$$

where $P(\xi)$ is a Gaussian distribution with zero mean, corresponding to the thermal probability distribution for the momentum of the environment particle. In arriving at these Kraus operators we have been guided by simplicity, in selecting $\hat{\sigma}$ to depend on the smallest possible number of parameters, and by the dynamics of the associated classical problem. Other forms of the Kraus operators are possible, of course, but selecting one of these possibilities will require a more detailed understanding of the collision process.

4. Quantum Brownian motion

In Brownian motion the particle is very much larger than the molecules of the surrounding medium that are responsible for its motion ($M \gg m$) and the collisions occur at a very great rate. This leads us to consider the continuous measurement limit in which $R \to \infty$ such

that $R(1-K) \approx 2Rm/M = \gamma$ and $R/\bar{n}$ are constants. In this limit our master equation becomes [23]

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}\left[\frac{\hat{p}^2}{2M}, \hat{\rho}\right] - \mathrm{i}\frac{\gamma}{2\hbar}\left[\hat{x}, \{\hat{p}, \hat{\rho}\}\right] - \frac{D_{pp}}{\hbar^2}\left[\hat{x}, \left[\hat{x}, \hat{\rho}\right]\right] - \frac{D_{xx}}{\hbar^2}\left[\hat{p}, \left[\hat{p}, \hat{\rho}\right]\right], \tag{28}$$

where

$$\begin{aligned} D_{pp} &= \gamma\left(Mk_{\mathrm{B}}T + \frac{m}{M}\Delta_\sigma p^2\right) + \frac{R\hbar^2}{8\Delta_\sigma x^2} \\ D_{xx} &= \frac{R\hbar^2}{8\Delta_\sigma p^2}. \end{aligned} \tag{29}$$

It is interesting to compare this master equation with (6) and compare the dynamics with the classical moment equations (4). Our master equation differs from (6) in two ways: the first is that the coefficient of $[\hat{x}, [\hat{x}, \hat{\rho}]]$ in (6) has additional coefficients and the second is the introduction of a double momentum-commutator term proportional to $[\hat{p}, [\hat{p}, \hat{\rho}]]$. These are responsible for differences between the classical and quantum equations of motion for the mean square position and momentum.

The evolution equations for the moments of position and momentum, derived from (28), are

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{x}\rangle = \frac{\langle\hat{p}\rangle}{M} \\ &\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{p}\rangle = -\gamma\langle\hat{p}\rangle \\ &\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{x}^2\rangle = \frac{2}{M}\frac{\langle\{\hat{x}, \hat{p}\}\rangle}{2} + \frac{R\hbar^2}{4\Delta_\sigma p^2} \\ &\frac{\mathrm{d}}{\mathrm{d}t}\frac{\langle\{\hat{x}, \hat{p}\}\rangle}{2} = \frac{\langle\hat{p}^2\rangle}{M} - \gamma\frac{\langle\{\hat{x}, \hat{p}\}\rangle}{2} \\ &\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{p}^2\rangle = -2\gamma\langle\hat{p}^2\rangle + 2\gamma\left(Mk_{\mathrm{B}}T + \frac{m}{M}\Delta_\sigma p^2\right) + \frac{R\hbar^2}{4\Delta_\sigma x^2}. \end{aligned} \tag{30}$$

Solution of these gives the steady-state mean kinetic energy

$$\begin{aligned} \frac{\langle\hat{p}^2(\infty)\rangle}{2M} &= \frac{k_{\mathrm{B}}T}{2} + \frac{m}{2M^2}\Delta_\sigma p^2 + \frac{R\hbar^2}{16M\gamma\Delta_\sigma x^2} \\ &= \frac{k_{\mathrm{B}}T}{2} + \frac{m}{2M^2}\Delta_\sigma p^2 + \frac{\hbar^2}{32m\Delta_\sigma x^2}, \end{aligned} \tag{31}$$

which clearly includes corrections to the classically expected $k_{\mathrm{B}}T/2$. The first of these can be understood as a low temperature or 'zero-point' energy contribution. To see this we note that a collision is accompanied by a change in the momentum, ζ, of the environment particle:

$$\delta\zeta \approx -2\zeta + \frac{2m}{M}p \tag{32}$$

and the momentum measurement must be effected by measuring this quantity. Our ability to resolve differences is set by the intrinsic quantum or zero-point momentum fluctuations:

$$\Delta_\sigma p^2 = \left(\frac{M}{m}\right)^2\langle\hat{\zeta}^2\rangle_0. \tag{33}$$

It follows that

$$\frac{m}{2M^2}\Delta_\sigma p^2 = \frac{\langle\hat{\zeta}^2\rangle_0}{2m}, \tag{34}$$

which is the zero-point kinetic energy of the environment particle. Hence we can interpret the first two terms in (31) as quantum equipartition of energy between the Brownian particle and those forming its environment.

The remaining contribution, $\hbar^2/(32m\Delta_\sigma x^2)$, has a more subtle interpretation which is most readily appreciated by reference to the last of the moment equations (30). It arises from the term $R\hbar^2/(4\Delta_\sigma x^2)$ and is a consequence of Heisenberg's uncertainty principle: if each collision localizes our particle to a region of width $\Delta_\sigma x$ then there is an associated induced momentum spreading of at least $\hbar/(2\Delta_\sigma x)$. The collisions occur at rate R and so we can interpret this contribution to the steady-state kinetic energy as a consequence of the effective localization of the particle by repeated measurement. A precisely analogous term, $R\hbar^2/(4\Delta_\sigma p^2)$, inducing spreading in the particle position occurs in the equation for $\langle\hat{x}^2\rangle$. It might seem strange, however, that the steady-state kinetic energy is at odds with what would have been expected on the basis of equipartition of energy (even if we include the 'zero-point' contribution). We should remember, however, that arguments based on equipartition apply for weakly interacting systems, but that our system is a free particle so the only energy scale is set by the interaction strength itself and we have no criterion for 'weak' interactions. For strongly interacting systems, of course, the interaction itself induces a change in the energy and this may be the microscopic origin of the final term in (31). In the absence of a microscopic model, however, this must remain a speculation. The steady-state diffusion of the particle's position is modified from its classical form (5), both by the change in the steady-state kinetic energy and by the spreading of position induced by the effective momentum measurement performed by each collision:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{x}^2\rangle\right|_{t\to\infty} = \frac{2\langle\hat{p}^2(\infty)\rangle}{M^2\gamma} + \frac{R\hbar^2}{4\Delta_\sigma p^2}. \tag{35}$$

The master equation obtained by consideration of the classical dynamics (6) is unsatisfactory because it can lead to negative probabilities at short times. Even if these do not arise for the relevant initial state, the fact that they can arise at all makes (6) unacceptable. It remains for us to show that this problem is not encountered for our master equation (28) and to explain why not. Mathematically, the positivity of the probabilities is ensured by the fact that (28) is of so-called Lindblad form [20, 33, 34]. It is more appealing, however, to examine the rate of change of the probability for remaining in the initial state, $|\psi\rangle$, as calculated in (7). For our master equation (28) we find

$$\left.\langle\psi|\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t}|\psi\rangle\right|_{t=0} = \frac{\gamma}{2} - \frac{2D_{pp}}{\hbar^2}\Delta x^2(0) - \frac{2D_{xx}}{\hbar^2}\Delta p^2(0). \tag{36}$$

A problem occurs if this derivative is positive, that is if we can find an initial state for which $\Delta x(0)$ and $\Delta p(0)$ are both sufficiently small. Here, of course, Heisenberg's uncertainty principle comes into play in that all states must satisfy the inequality $\Delta x(0)\Delta p(0) \geqslant \hbar/2$ so that $\Delta x(0)$ and $\Delta p(0)$ cannot both be arbitrarily small. By requiring the derivative (36) to be less than or equal to zero for all possible states (and hence for all possible uncertainties $\Delta x(0)$ and $\Delta p(0)$) we are led to the requirement that

$$D_{xx}D_{pp} \geqslant \left(\frac{\hbar\gamma}{4}\right)^2. \tag{37}$$

Inspection of (29) shows that this inequality is satisfied and so our master equation does not exhibit the unphysical behaviour of (6). Clearly *both* types of diffusion, momentum and position, are needed in a quantum theory. It is interesting to ask, however, which of the processes we have identified is fundamentally responsible for satisfying the inequality (37). We can make the product $D_{xx}D_{pp}$ as small as possible by considering the low temperature

behaviour, $T \approx 0$, and by supposing that the effective position measurements are very imprecise so that $\Delta_\sigma x^2 \to \infty$. In this limit we find

$$\begin{aligned} D_{pp} &= \gamma \frac{m}{M} \Delta_\sigma p^2 \\ D_{xx} &= \frac{R\hbar^2}{8\Delta_\sigma p^2} = \gamma \frac{M}{m} \frac{\hbar^2}{16\Delta_\sigma p^2}, \end{aligned} \tag{38}$$

so that (37) is satisfied as an equality. We can conclude that it is the coexistence of the zero-point energy contribution to D_{pp} and of the position diffusion, D_{xx}, caused by the effective momentum measurement, that ensures the physical behaviour of our quantum theory of friction.

5. Conclusion

The classical theory of friction does not transfer in any simple way into the quantum regime. This is a problem for the emerging quantum information processing technologies in which environmentally induced decoherence is a key factor in limiting both performance and scalability. Our approach to describing friction is, in a sense, based on ideas in quantum information theory in that it emphasizes the role of measurement, of information extraction, on the dynamics of the particle.

The change in state associated with a generalized measurement is most naturally described in the language of an effect [31]. If we assume that the measurements occur instantaneously and at a characteristic rate R then we are led to a master equation (15) for a general system subjected to 'continuous' measurements [32]. Choosing a measurement of the particle's position and momentum and incorporating simple classical ideas of collision dynamics leads us to our master equation for friction (19). Brownian motion is associated with a particle with a mass greatly exceeding that of the molecules forming the surrounding medium. In this limit our master equation simplifies to (28), which differs from the form derived from classical arguments (6) by a modification of the coefficient of the double position-commutator and by the introduction of a double momentum-commutator. These lead to modifications of the steady-state kinetic energy and to the rate of diffusion of the particle's position. They are intrinsically quantum in nature and are associated with the 'zero-point' motion of the molecules forming the medium and, through Heisenberg's uncertainty principle, with the accuracy of the effective position and momentum measurements associated with each collision.

The work presented here and in [23] has established a plausible link between generalized measurements of position and momentum and the phenomenon of environment induced friction. A fully quantum microscopic model and derivation is, however, highly desirable. With such a model we should be able to determine the 'zero-point' energy contribution and to determine the variances, $\Delta_\sigma x^2$ and $\Delta_\sigma p^2$, which limit the accuracy of the effective measurements. It will also enable us to derive the friction coefficient γ and to determine its temperature dependence. An elementary argument based on kinetic theory suggests that at high temperatures $R \propto \sqrt{T}$ and as a consequence γ will also be proportional to $\sqrt{T}$. In addition, D_{xx} and D_{pp} will acquire temperature dependences which are quite different from those found by other authors. At lower temperatures, however, a quantum correction to this is to be anticipated. Friction affects particles moving in potentials and it is important to determine how the presence of a potential will change our theory. Damped harmonic oscillators, for example, have a long been a familiar component of quantum optics [10–12]. The methods used to describe them, however, rely on the existence of a well defined frequency so that the oscillator is always weakly damped. A satisfactory quantum theory of friction, however, should allow us to describe critical damping and overdamping of the oscillator.

Finally we should note that because our theory is Markovian and has the required Lindblad form we can model the dynamics of a single particle by performing stochastic wavefunction

simulation [35, 36]. This allows us to describe both the quantum dynamics of the particle and, at the same time, the continuous measurement record. It has been suggested that such systems are important in the emergence of classical behaviour, including chaos [37] from quantum systems. We will return to this point elsewhere.

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